

Algebraic Properties of Stochastic Effectivity Functions

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Dedicated to Professor Prakash Panangaden on the occasion of his 60th birthday.

Abstract

Effectivity functions are the basic formalism for investigating the semantics game logic. We discuss algebraic properties of stochastic effectivity functions, in particular the relationship to stochastic relations, morphisms and congruences are defined, and the relationship of abstract logical equivalence and behavioral equivalence is investigated.

1 Introduction

This paper investigates some algebraic properties of stochastic effectivity functions, which have been used as the basic formalism for interpreting game logic, and which show some interesting relationships to non-deterministic labelled Markov processes and to stochastic relations, i.e., to Markov transition systems. Before entering into the discussion of the stochastic branch of this family of functions, it is interesting and illuminating to have a look at the evolution of these functions, so let us start with some historical remarks.

Historical Remarks. Effectivity functions were first systematically investigated in the area of Social Choice, and here in particular for the modeling of voting systems [27, Chapter 7.2]. Moulin models the outcome of cooperative voting through a binary relation $\mathbf{E} \subseteq \mathcal{P}(\mathbf{N}) \times \mathcal{P}(\mathbf{A})$ between coalitions of voters and subsets of outcomes, where a coalition is just a subset of the entire population. Here \mathbf{N} is the set of voters, \mathbf{A} the set of outcomes, and \mathcal{P} denotes the power set. If $\mathbf{T} \mathbf{E} \mathbf{B}$, coalition \mathbf{T} is said to be *effective* for the subset \mathbf{B} of outcomes, thus coalition \mathbf{T} can force an outcome in \mathbf{B} . Among others, Moulin postulates that if \mathbf{T} is effective for \mathbf{B} , then $\mathbf{N} \setminus \mathbf{T}$ must not be effective for $\mathbf{S} \setminus \mathbf{B}$. Some examples (unanimity with status quo, veto functions) illustrate the approach.

Generalizing this in their work on Social Choice, Abdou and Keiding [1] define effectivity functions as special cases of conditional game forms. Such a *conditional game form* is a map $\mathbf{E} : \mathcal{P}(\mathbf{N}) \rightarrow \mathcal{P}(\mathcal{A} \setminus \{\emptyset\})$, where \mathbf{N} and \mathbf{A} are as above, \mathcal{A} is a subset of $\mathcal{P}(\mathbf{A})$ with $\emptyset \in \mathcal{A}$, $\mathbf{A} \in \mathcal{A}$. The family of all closed sets in a topological space or of all measurable subsets in a measurable space are mentioned as examples. If \mathbf{T} is a coalition, $\mathbf{B} \in \mathbf{E}(\mathbf{T})$ models that coalition \mathbf{T} can

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force an outcome in B . Thus the notion of effectivity is the same as in Moulin's proposal. Based on this, a first simple and general form of effectivity function is defined. A conditional game form $E : \mathcal{P}(N) \rightarrow \mathcal{P}(\mathcal{A} \setminus \{\emptyset\})$ is called an *effectivity function* iff $A \in E(T)$ for all non-empty coalitions T , and if $E(N) = \mathcal{A} \setminus \{\emptyset\}$. Thus a non-empty coalition can achieve something, and the community N has all options to choose from.

The functions discussed so far do not make an assumption on monotonicity; for the purposes of the present paper, however, monotone effectivity functions are of interest. This property appears to be natural, given the interpretation of effectivity functions: If an outcome in B can be forced by coalition T , then an outcome in each super set $B' \supseteq B$ can be forced by this coalition as well; if this is true for all coalitions, function E is called *monotone*.

The neighborhood relations used here are taken from the minimal models discussed in modal logics [4, Chapter 7.1], serving as basic mechanism for models which are more general than Kripke models. The association of the effectivity functions sketched here to a very similar notion investigated in economics is discussed in the survey paper [32, Section 2.3].

Game Logic. Parikh [29], and later Pauly [30] propose interpreting game logic through a neighborhood model. Assign to each primitive game g and each player $\{1, 2\}$ a neighborhood relation $N_g^{(i)} \subseteq S \times \mathcal{P}(S)$ ($i = 1, 2$) with the understanding that $sN_g^{(i)}X$ indicates player i having a strategy in state s to force a state in $X \subseteq S$. Here S is the set of states over which the game is interpreted. The fact that $sN_g^{(i)}X$ is sometimes described by saying that player i is effective for X (with game g in state s). It is desirable that $sN_g^{(i)}X$ and $X \subseteq X'$ imply $sN_g^{(i)}X'$ for all states s . We assume in addition that the game is *determined*, i.e., that exactly one of the players has a winning strategy. Thus $X \subseteq S$ is effective for player 1 in state s if and only if $S \setminus X$ is not effective for player 2 in that state. Consequently,

$$sN_g^{(2)}X \Leftrightarrow \neg(sN_g^{(1)}S \setminus X), \quad (1)$$

which in turn implies that we only have to cater for player 1. We will omit the superscript from the neighborhood relation N_g . Define a map $S \rightarrow \mathcal{P}(\mathcal{P}(S))$, again denoted by N_g , upon setting $N_g(s) := \{X \subseteq S \mid sN_gX\}$, then $N_g(s)$ is an upper closed subset of $\mathcal{P}(S)$ for all $s \in S$ from which relation N_g can be recovered. This function is called the *effectivity function* associated with relation N_g . From N_g another map $\check{N}_g : \mathcal{P}(S) \rightarrow \mathcal{P}(S)$ is obtained upon setting

$$\check{N}_g(A) := \{s \in S \mid sN_gA\} = \{s \in S \mid A \in N_g(s)\}.$$

Thus state s is an element of $\check{N}_g(A)$ iff the first player has a strategy force the outcome in A when playing g in s . The operations on games can be taken care of through this family of maps, e.g., one sets recursively for the first player

$$\check{N}_{g_1 \cup g_2}(A) := \check{N}_{g_1}(A) \cup \check{N}_{g_2}(A), \quad (2)$$

$$\check{N}_{g_1; g_2}(A) := (\check{N}_{g_1} \circ \check{N}_{g_2})(A), \quad (3)$$

$$\check{N}_{g^*} := \bigcup_{n \geq 0} \check{N}_{g^n}(A), \quad (4)$$

with $g_1 \cup g_2$ denoting the game in which the first player chooses from games g_1, g_2 , the game $g_1; g_2$ plays g_1 first, then g_2 , and g^* is the indefinite iteration of game g . This refers only to player 1, player 2 is accommodated through $A \mapsto S \setminus \check{N}_g(S \setminus A)$ by (1), since the game is determined. Pauly [30, Section 6.3] discusses the important point of determinacy of games and relates it briefly to the discussion in set theory [23, Section 33], [22, Section 12.3] or [24, Section 20].

The maps \check{N}_g serve in Parikh's original paper as a basis for defining the semantics of game logic. It turns out to be convenient for the present paper to go back, and to use effectivity functions as maps to upper closed subsets. When interpreting game logic probabilistically, however, constructions (2) – (4) are fairly meaningless, because one cannot talk about, e.g., the union of two probabilities. Hence another path had to be travelled, which may be seen as a technical generalization of the proposal [12] for interpreting propositional dynamic logic.

Requirements. The paper [11] proposes an approach through stochastic effectivity functions. This variant assigns to each state sets of probability distributions, to be specific, each state is assigned an upward closed set of measurable sets of these distributions. This property is inherited from the effectivity functions discussed above.

But let us have a look at other requirements for a stochastic effectivity function. They should generalize stochastic relations in the sense that each stochastic relation generates a stochastic effectivity function in a fairly natural way; this requirement will make it possible to consider the interpretation of modal logics through stochastic Kripke models as interpretations through stochastic effectivity functions, hereby enabling us to compare — or contrast, as the case may be — results for these formalisms. We want also to state that morphisms for stochastic relations are morphisms for the associated effectivity functions. In addition, we want to have some notion of measurability for these functions. Consequently, the set of all distributions has to be made a measurable space. This raises immediately the question of measurability of the effectivity function proper, which would require a measurable structure on the upward closed subsets of the measurable sets of the set of distributions over the state space.

The design of non-deterministic labelled Markov processes proceeds along this path: the set of measurable subsets of the measures on a measurable space is equipped with a σ -algebra (akin to defining a topology on the space of all closed subsets of a topological space), and measurability is defined through this σ -algebra [5], the *hit σ -algebra*. It turns out that one of the important issues here is composability of effectivity functions for the purpose of catering for the composition of games (or, more general, of actions). This is difficult to achieve with the concept of hit measurability, and it will be taken care of by the concept of t-measurability, which is introduced below.

Let us briefly discuss the issue of composability. Bind actions γ and δ to effectivity functions P_γ and P_δ ; we want to model action $\gamma; \delta$, the sequential execution of γ and δ . Given a measurable set A , the sets $\beta(A, r) := \{\mu \mid \mu(A) > r\}$ collect all the evaluations for A , where $r \in [0, 1]$, so $\mu \in \beta(A, r)$ means that event A happens with probability greater than r , if the probability is governed by μ . Suppose that we know the set $Q_\delta(A, r)$ of all states in which an evaluation of the measurable set A can be achieved through action δ . We pick all state distributions the expectation of which over $Q_\delta(A, r)$ is greater than a threshold value q , hence

we look at the set

$$\{\nu \mid \int_0^1 \nu(Q_\delta(A, r)) > q\} \quad (5)$$

If this set is an element of $P_\gamma(s)$, then we say that the composed action $\gamma; \delta$ can achieve a distribution which evaluates at event A with a probability greater than q .

In order to get this slightly involved machinery going, however, we have to make sure that the set described in (5) is actually a measurable of measures. Thus we need a suitable concept of measurability, which we call *t-measurability* and which is formulated below.

Overview. This paper discusses stochastic effectivity functions from a general point of view, pointing out some algebraic properties which might be helpful when using these functions for interpreting logics. We introduce these functions after a careful discussion of the underlying concept of measurability, and after comparing this concept to the one proposed with a model for non-deterministic randomness. The relationship to stochastic relations, or, as they are called sometimes, sub probabilistic Markov kernels is discussed at length, and it is shown that stochastic effectivity functions form — together with non-deterministic Markov processes as their close cousins — a true generalization of these kernels. This applies as well to morphisms and, in consequence, to congruences, which are introduced and discussed. When interpreting logics through a particular class of models, the question of expressivity becomes important, the main issues being behavioral and logical equivalence, with bisimilarity being another main actor in this play. We discuss the question of behavioral and logical equivalence in the present paper, formulating these properties without having to go back to an underlying logic; this is done as in [8] in purely algebraic terms through morphisms and congruences. We give criteria under which logically equivalent effectivity functions are behaviorally equivalent.

Before we set out to discuss all these properties, we remind the reader of some properties of measurable spaces, providing some tools on the way. We also investigate some interesting properties of final measurable surjections; they turn out to be helpful when it comes to scrutinize σ -algebras of invariant sets and spaces of subprobabilities. These maps turn out to be a very convenient means of transporting measures faithfully, which becomes of interest when investigating congruences.

2 Basic Definitions

The reader is briefly reminded of some notions and constructions from measure theory, including the famous π - λ -Theorem and Choquet's representation of integrals as areas; these two tools are used all over the paper. We also introduce sets which are invariant under an equivalence relation together with the corresponding σ -algebra of invariant sets. As a special case of equivalence relations we introduce tame relations, which are equivalence relations that are compatible with quantitative measurements.

2.1 Measurability

First we fix some notations. A measurable space (S, \mathcal{S}) is a set S with a σ -algebra \mathcal{S} , i.e., $\mathcal{S} \subseteq \mathcal{P}(S)$ is a Boolean algebra which is closed under countable unions. Here $\mathcal{P}(S)$ is the power set of S . Given $\mathcal{S}_0 \subseteq \mathcal{P}(S)$, denote by

$$\sigma(\mathcal{S}_0) := \bigcap \{ \mathcal{T} \mid \mathcal{S}_0 \subseteq \mathcal{T}, \mathcal{T} \text{ is a } \sigma\text{-algebra} \}$$

the smallest σ -algebra containing \mathcal{S}_0 (the set for which the intersection is constructed is not empty, since it contains $\mathcal{P}(S)$). If S is a topological space with topology τ , then the elements of $\sigma(\tau)$ are called the *Borel sets* of S ; the σ -algebra $\sigma(\tau)$ is usually denoted by $\mathcal{B}(S)$.

Given two measurable spaces (S, \mathcal{S}) and (T, \mathcal{T}) , the product space $(S \times T, \mathcal{S} \otimes \mathcal{T})$ has the Cartesian product $S \times T$ as a carrier set, the product σ -algebra

$$\mathcal{S} \otimes \mathcal{T} := \sigma(\{A \times B \mid A \in \mathcal{S}, B \in \mathcal{T}\})$$

is the smallest σ -algebra on $S \times T$ which contains all measurable rectangles $A \times B$ with $A \in \mathcal{S}$ and $B \in \mathcal{T}$. Define for $E \subseteq S \times T$

$$\begin{aligned} E^s &:= \{t \in T \mid \langle s, t \rangle \in E\} && \text{(vertical cut),} \\ E_t &:= \{s \in S \mid \langle s, t \rangle \in E\} && \text{(horizontal cut),} \end{aligned}$$

then $E^s \in \mathcal{T}$ for all $s \in S$, and $E_t \in \mathcal{S}$ for all¹ $t \in T$, provided $E \in \mathcal{S} \otimes \mathcal{T}$. The converse does not hold: Let S have a cardinality strictly larger than that of the continuum, then $\mathcal{P}(S) \otimes \mathcal{P}(S)$ is a proper subset of $\mathcal{P}(S \times S)$, because $\Delta := \{\langle s, s \rangle \mid s \in S\} \notin \mathcal{P}(S) \otimes \mathcal{P}(S)$ [19, Exercise 21.20]. On the other hand, $\Delta_s = \{s\} = \Delta^s \in \mathcal{P}(S)$ for all $s \in S$, so we cannot conclude that a set is product measurable, provided all its cuts are measurable.

If (S, τ) and (T, ϑ) are topological spaces, then the Borel sets $\mathcal{B}(\tau \times \vartheta)$ of the product topology may properly contain the product $\mathcal{B}(\tau) \otimes \mathcal{B}(\vartheta)$. If, however, both spaces are Hausdorff and ϑ has a countable basis, then $\mathcal{B}(\tau \times \vartheta) = \mathcal{B}(\tau) \otimes \mathcal{B}(\vartheta)$ [3, Lemma 6.4.2]. In particular, the Borel sets of the product of two Polish spaces are the product of the Borel sets of the components (a *Polish space* is a topological space which has a countable base and for which a complete metric exists). The same applies to analytic spaces (an *analytic space* is a separable metric space which is the continuous image of a Polish space), since the topology of these spaces is also countably generated.

In summary, the observation on products mentioned above suggests that we have to exercise particular care when working with the product of two measurable spaces, which carry a topological structure as well.

Given the measurable spaces (S, \mathcal{S}) and (T, \mathcal{T}) , a map $f : S \rightarrow T$ is said to be *\mathcal{S} - \mathcal{T} -measurable* iff $f^{-1}[D] \in \mathcal{S}$ for all $D \in \mathcal{T}$. Call the measurable map $f : (S, \mathcal{S}) \rightarrow (T, \mathcal{T})$ *final* iff \mathcal{T} is the final σ -algebra with respect to f and \mathcal{S} . Hence f is final iff \mathcal{T} is the largest σ -algebra \mathcal{C} on T such that $f^{-1}[\mathcal{C}] := \{f^{-1}[C] \mid C \in \mathcal{C}\} \subseteq \mathcal{S}$ holds, so that $\mathcal{T} = \{B \subseteq T \mid f^{-1}[B] \in \mathcal{S}\}$. Hence we may conclude from $f^{-1}[B] \in \mathcal{S}$ that $B \in \mathcal{T}$. An equivalent formulation for finality of f is that

¹The notation of indicating the horizontal cut through an index conflicts with indexing, but it is customary, so we will be careful to make sure which meaning we have in mind.

a map $g : T \rightarrow U$ is \mathcal{T} - \mathcal{U} -measurable if and only if $g \circ f : S \rightarrow U$ is \mathcal{S} - \mathcal{U} -measurable, whenever (U, \mathcal{U}) is a measurable space. Measurability of real valued maps always refers to the Borel sets on the reals, hence $f : S \rightarrow \mathbb{R}$ is measurable iff $\{s \in S \mid f(s) \bowtie q\} \in \mathcal{S}$ for each rational number q , with \bowtie as one of the relations $\leq, <, \geq, >$.

Measurable spaces with measurable maps as morphisms form a category, which, however, does not have an extra symbol assigned to it in the present paper.

Let ρ be an equivalence relation on S . Call $A \subseteq S$ an ρ -invariant set iff A is the union of ρ -classes, equivalently, iff $s \in A$ and $s \rho s'$ implies $s' \in A$. Then

$$\Sigma(\rho, \mathcal{S}) := \{A \in \mathcal{S} \mid A \text{ is } \rho\text{-invariant}\}$$

denotes the σ -algebra of ρ -invariant measurable subsets of S . As usual,

$$\ker(f) := \{\langle s, s' \rangle \mid f(s) = f(s')\}$$

is the *kernel* of f .

We write $\mathbb{S}(S, \mathcal{S})$ for the set of all subprobability measures on the measurable space (S, \mathcal{S}) . This space is made a measurable space upon taking as a σ -algebra

$$w(\mathcal{S}) := \sigma(\{\beta_{(S, \mathcal{S})}(A, \bowtie q) \mid A \in \mathcal{S}, q \in [0, 1]\}) \quad (6)$$

Here

$$\beta_{(S, \mathcal{S})}(A, \bowtie q) := \{\mu \in \mathbb{S}(S, \mathcal{S}) \mid \mu(A) \bowtie q\}$$

is the set of all subprobabilities on (S, \mathcal{S}) which evaluate on the measurable set A as $\bowtie q$, where \bowtie is one of the relations $\leq, <, \geq, >$. This σ -algebra is sometimes called the *weak- \ast - σ -algebra*.

A morphism $f : (S, \mathcal{S}) \rightarrow (T, \mathcal{T})$ in the category of measurable spaces induces a map $\mathbb{S}f : \mathbb{S}(S, \mathcal{S}) \rightarrow \mathbb{S}(T, \mathcal{T})$ upon setting

$$(\mathbb{S}f)(\nu)(B) := \nu(f^{-1}[B])$$

for $B \in \mathcal{B}(T, \mathcal{T})$; as usual, $\mathbb{S}f$ is sometimes written as $\mathbb{S}(f)$. Because $(\mathbb{S}f)^{-1}[\beta_T(B, \bowtie q)] = \beta_S(f^{-1}[B], \bowtie q)$, this map is $w(\mathcal{S})$ - $w(\mathcal{T})$ -measurable as well. Thus \mathbb{S} is an endofunctor on the category of measurable spaces with measurable maps as morphisms; in fact, it is the functorial part of a monad which is sometimes called the Giry monad [16], for a slight extension see [7].

From now on, we will not write down explicitly the σ -algebra \mathcal{S} of a measurable space (S, \mathcal{S}) , unless there is good reason to do so. Furthermore the space $\mathbb{S}(S)$ of all subprobabilities will be understood to carry the weak- \ast - σ -algebra $w(\mathcal{S})$ **always**. We will write Σ_ρ for $\Sigma(\rho, \mathcal{S})$, and $\Sigma_{\ker(f)}$ will be abbreviated as Σ_f .

Definition 2.1 *Given two measurable spaces S and T , a stochastic relation (or sub Markov kernel) $K : S \rightsquigarrow T$ from S to T is a measurable map $S \rightarrow \mathbb{S}(T)$.*

$K : S \rightsquigarrow T$ is a stochastic relation iff these conditions hold

1. $K(s)$ is for each $s \in S$ a subprobability measure on the σ -algebra \mathcal{T} of T .
2. For each $D \in \mathcal{T}$, the map $s \mapsto K(s)(D)$ is measurable.

This characterization is well known. A stochastic relation $K : S \rightsquigarrow S$ models probabilistic transitions: $K(s)(C)$ is interpreted as the probability that the next state is a member of D after making a transition from s ; if $K(s)(S) < 1$, the event that there is no next state may occur with positive probability.

It can be shown that stochastic relations are the Kleisli morphisms for the Giry monad [16].

2.2 Some Indispensable Tools

We post here for the reader's convenience some measure theoretic tools which will be used all over. Fix a set S .

Dynkin's π - λ -Theorem. This technical tool is most useful when it comes to determine the σ -algebra generated by a family of sets [24, Theorem 10.1].

Proposition 2.2 *Let \mathcal{A} be a family of subsets of S that is closed under finite intersections. Then $\sigma(\mathcal{A})$ is the smallest family of subsets containing \mathcal{A} which is closed under complementation and countable disjoint unions.* \square

Choquet's Representation. The following condition on product measurability and an associated integral representation attributed to Choquet is used [3, Corollary 3.4.3]. Assume that (S, \mathcal{S}) is a measurable space.

Theorem 2.3 *Let $f : S \rightarrow \mathbb{R}_+$ be measurable and bounded, then*

$$C_{\bowtie}(f) := \{\langle s, r \rangle \in S \times \mathbb{R}_+ \mid f(s) \bowtie r\} \in \mathcal{S} \otimes \mathcal{B}(\mathbb{R}_+). \quad (7)$$

If μ is a σ -finite measure on S , then

$$\int_S f(s) \mu(dx) = \int_0^\infty \mu(\{s \in S \mid f(s) > t\}) dt = (\mu \otimes \lambda)(C_{>}(f)). \quad (8)$$

with $\mu \otimes \lambda$ as the product of μ with Lebesgue measure λ . \square

For S an interval in \mathbb{R} , the set $C_{>}(f) = \{\langle s, t \rangle \in S \times \mathbb{R}_+ \mid 0 \leq t < f(s)\}$ may be visualized as the area between the x -axis and the graph of f . Hence formula (8) specializes to the Riemann integral, if $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is Riemann integrable, and μ is also Lebesgue measure.

2.3 Tame Relations

A σ -algebra $\mathcal{A} \subseteq \mathcal{S}$ on the measurable space (S, \mathcal{S}) induces an equivalence relation $\rho_{\mathcal{A}}$ upon setting

$$s \rho_{\mathcal{A}} s' :\Longleftrightarrow [\forall A \in \mathcal{A}_0 : s \in A \text{ iff } s' \in A] \quad (9)$$

for some generator \mathcal{A}_0 of \mathcal{A} (\mathcal{A}_0 may be \mathcal{A} , of course). It is easy to see that each element of \mathcal{A} is $\rho_{\mathcal{A}}$ -invariant. But we do not have necessarily $\Sigma_{\rho_{\mathcal{A}}} = \mathcal{A}$: Take for example S as the reals \mathbb{R} , where the Borel sets $\mathcal{B}(\mathbb{R})$ are taken as the σ -algebra, and take \mathcal{Y} as the countable-cocountable sub- σ -algebra of $\mathcal{B}(\mathbb{R})$, then $\rho_{\mathcal{Y}}$ is the identity, and $\Sigma_{\rho_{\mathcal{Y}}} = \mathcal{B}(\mathbb{R})$.

Definition 2.4 *Given an equivalence relation ρ and a subset $\mathcal{A} \subseteq S$ of the measurable sets of S , we call ρ exact with \mathcal{A} iff $\Sigma_{\rho} = \sigma(\mathcal{A})$.*

Thus \mathcal{A} generates *exactly* the invariant measurable sets of ρ , if ρ is exact with \mathcal{A} ; this notion will be helpful below. It is easy to see that \mathcal{A} determines ρ as in (9). Taking \mathcal{Y} as above, we see that $\rho_{\mathcal{Y}}$ is not exact with \mathcal{Y} ; it is, however, exact with the open sets τ or the intervals \mathcal{I} of \mathbb{R} , because $\Sigma_{\rho_{\mathcal{Y}}} = \mathcal{B}(\mathbb{R}) = \sigma(\tau) = \sigma(\mathcal{I})$.

The set S/ρ of all equivalence classes is endowed with the final σ -algebra S/ρ with respect to the factor map $\eta_{\rho} : s \mapsto [s]_{\rho}$, i.e., the largest σ -algebra rendering η_{ρ} measurable. Hence

$$S/\rho = \{C \subseteq S/\rho \mid \eta_{\rho}^{-1}[C] \in \Sigma\}.$$

It follows that $\eta_{\rho}[B] \in S/\rho$ whenever $B \in \Sigma_{\rho}$, because $B = \eta_{\rho}^{-1}[\eta_{\rho}[B]]$ on account of the invariance of B .

The first part of the following statement is obvious.

Lemma 2.5 *Let $f : S \rightarrow T$ be a measurable map, then $\ker(f)$ is exact with Σ_f . If f is final, then $\{f^{-1}[E] \mid E \in \mathcal{T}\} = \Sigma_f$.*

Proof 1. If $E \in \mathcal{T}$, we know that $f^{-1}[E] \in \Sigma$. Since $f^{-1}[E]$ is $\ker(f)$ -invariant, we conclude that $\{f^{-1}[E] \mid E \in \mathcal{T}\} \subseteq \Sigma_f$.

2. For the other inclusion, decompose f as $\tilde{f} \circ \eta_{\ker(f)}$, then $\tilde{f} : S/\ker(f) \rightarrow T$ is injective, and, since the domain carries a final σ -algebra, it is measurable. Because \tilde{f} is injective, the inverse $\tilde{f}^{-1} : \mathcal{P}(T) \rightarrow \mathcal{P}(S/\ker(f))$ is onto. Now let $D \in \Sigma_f$, then $\eta_{\ker(f)}[D] \in S/\ker(f)$, because $D = \eta_{\ker(f)}^{-1}[\eta_{\ker(f)}[D]]$, since D is $\ker(f)$ -invariant. Because \tilde{f}^{-1} is onto, we find some $E \subseteq T$ with $\eta_{\ker(f)}[D] = \tilde{f}^{-1}[E]$, and we conclude $D = \eta_{\ker(f)}^{-1}[\eta_{\ker(f)}[D]] = \eta_{\ker(f)}^{-1}[\tilde{f}^{-1}[E]] = f^{-1}[E]$, so that $E \in \mathcal{T}$, since in particular $D \in \Sigma$. This implies $\Sigma_f \subseteq \{f^{-1}[E] \mid E \in \mathcal{T}\}$. \square

Let us briefly mention an important special case. Assume that S and T are Polish, then each Borel measurable and surjective map $f : S \rightarrow T$ is final [10, Proof of Lemma 1.7.10]. Hence final maps occur in a fairly natural way in a topological setting.

The following observations are helpful consequences. We first show that Σ_f and \mathcal{T} have the same structure as Boolean σ -algebras, if $f : S \rightarrow T$ is final and onto.

Corollary 2.6 *Let $f : S \rightarrow T$ be final and surjective. Then f^{-1} is an isomorphism of the σ -algebras Σ_f and \mathcal{T} .*

Proof Because f is onto, $f^{-1} : \mathcal{T} \rightarrow \Sigma_f$ is injective. Now let $E \in \Sigma_f$; we claim that $f^{-1}[f[E]] = E$. In fact, let $f(s) \in f[E]$, then there exists $s' \in E$ with $f(s) = f(s')$. Since E is $\ker(f)$ -invariant, we conclude $s \in E$, hence $f^{-1}[f[E]] \subseteq E$. The other inclusion is trivial. Because f is final, we conclude from $f^{-1}[f[E]] \in \Sigma_f \subseteq \Sigma$ that $f[E] \in \mathcal{T}$. Hence f^{-1} is onto as well, so it is

a bijection. Because f^{-1} is compatible with finite or countable Boolean operations, we have established that f^{-1} is the isomorphism we are looking for. \square

We could have used the induced map $f : \Sigma_f \rightarrow \mathcal{T}$ in the scenario above and then have shown that f is an isomorphism. The crucial observation is that $f[\bigcap_{i \in I} A_i] = \bigcap_{i \in I} f[A_i]$ holds for any family $(A_i)_{i \in I}$ in Σ_f . In fact, assume that $I \neq \emptyset$, and let $t \in \bigcap_{i \in I} f[A_i]$. Then there exists for each $j \in I$ an element $a_j \in A_i$ with $t = f(a_j)$. Because all A_i are $\ker(f)$ -invariant, one concludes that $a_j \in \bigcap_{i \in I} A_i$ for all $j \in I$, hence $t \in f[\bigcap_{i \in I} A_i]$; the other inclusion is trivial. In the same manner one sees that $f[S \setminus A] = \mathcal{T} \setminus f[A]$ for $A \in \Sigma_f$. Hence the direct image can be used for the proof of Corollary 2.6 as well. But usually the inverse image is more convenient to work with whenever measures are concerned.

From Corollary 2.6 we obtain as a consequence that the measure spaces $\mathbb{S}(S, \Sigma_f)$ and $\mathbb{S}(T, \mathcal{T})$ are isomorphic as measurable spaces. Quite apart of being of independent interest, we will use this observation for investigating subprobabilities on one space through those on the other space.

Corollary 2.7 *Let $f : S \rightarrow T$ be final and surjective. Then $\mathbb{S}(f) : \mathbb{S}(S, \Sigma_f) \rightarrow \mathbb{S}(T, \mathcal{T})$ is an isomorphism for the measurable spaces with the respective weak- $*$ - σ -algebras.*

Proof The inverse image map $f^{-1} : \mathcal{T} \rightarrow \Sigma_f$ is an isomorphism of the Boolean σ -algebras by Corollary 2.6. Thus, given $\nu \in \mathbb{S}(T, \mathcal{T})$, $\mu(C) := \nu(f[C])$ defines a subprobability measure on (S, Σ_f) with $\mathbb{S}(f)(\mu) = \nu$, because $\mathbb{S}(f)(\mu)(E) = \mu(f^{-1}[E]) = \nu(f[f^{-1}[E]]) = \nu(E)$. Thus $\mathbb{S}(f)$ is surjective; it is injective as well, since $\mathbb{S}(f)(\mu_1)(E) \neq \mathbb{S}(f)(\mu_2)(E)$ means that μ_1 and μ_2 differ on $f^{-1}[E] \in \Sigma_f$, hence are different members of $\mathbb{S}(S, \Sigma_f)$. The generators of the weak- $*$ - σ -algebras are in a bijective correspondence with each other, which implies that $\mathbb{S}(S, \Sigma_f)$ and $\mathbb{S}(T, \mathcal{T})$ are also isomorphic as measurable spaces. \square

If ρ is exact with \mathcal{A} , we have a handle on the elements of \mathbb{S}/ρ , albeit in a special situation. The characterization below is very similar to Corollary 2.6.5 in [10]. That statement deals with validity sets of the formulas of a negation free logic which is closed under finite conjunctions. The proof given there carries over easily to the situation at hand.

Lemma 2.8 *Let ρ be exact with \mathcal{A} , and assume that \mathcal{A} is closed under finite intersections. Then we have*

1. $\Sigma_\rho = \sigma(\mathcal{A})$,
2. $\mathbb{S}/\rho = \sigma(\{B \subseteq S/\rho \mid \eta_\rho^{-1}[B] \in \mathcal{A}\})$. \square

We require a slightly stronger condition on the equivalence relations we are dealing with, because we need to consider reals in the unit interval $[0, 1]$ as well. Define for this the equivalence relation $\rho \times \Delta$ on $S \times [0, 1]$ upon setting

$$\langle s, q \rangle (\rho \times \Delta) \langle s', q' \rangle \text{ iff } s \rho s' \text{ and } q = q'.$$

Let us call an equivalence relation ρ tame if the invariant sets of $\rho \times \Delta$ behave well. In descriptive set theory, countably generated equivalence relations on a Polish space are sometimes called tame (they are called *smooth* in [10]); the behavior of the present tame relations is modeled after them. The tame relations in the present paper are quite a different concept from the one discussed by Jacobs in [21].

Definition 2.9 *Call an equivalence relation ρ on the measurable space (S, \mathcal{S}) tame iff these conditions hold*

1. ρ is exact with some $\mathcal{A} \subseteq \mathcal{S}$,
2. $\rho \times \Delta$ is exact with $\{A \times I \mid A \in \mathcal{A}, I \in \mathcal{B}([0, 1])\}$

Thus tameness of ρ tells us that the invariant sets of $\rho \times \Delta$ can just be generated through Σ_ρ and $\mathcal{B}([0, 1])$, the invariant sets for Δ . Consequently, dealing with the invariant sets for $\rho \times \Delta$ becomes more practical through tameness ([11, Lemma 3.8]). Just for the record:

Lemma 2.10 *The equivalence relation ρ is tame iff $\Sigma_{\rho \times \Delta} = \Sigma_\rho \otimes \mathcal{B}([0, 1])$ holds.* \square

Thus we can characterize the factor space with respect to $\rho \times \Delta$ easily; for a proof see [11, Corollary 3.9].

Corollary 2.11 *Assume that ρ is tame, then $(S \otimes [0, 1])/\rho \times \Delta$ and $S/\rho \otimes [0, 1]$ are isomorphic as measurable spaces.* \square

We digress briefly and establish tameness in an interesting special case. Recall that a *smooth equivalence relation* ρ on S has a countable set $\mathcal{G} \subseteq \mathcal{S}$ such that $\Sigma_\rho = \sigma(\mathcal{G})$. Hence smooth equivalence relations are countably generated. A concise discussion of the properties of these relations can be found in [10, Section 1.7].

Proposition 2.12 *If ρ is smooth, and S is Polish, then ρ is tame.*

Proof Since S is Polish and ρ is countably generated, the factor space S/ρ is an analytic space [10, Proposition 1.7.5], so in particular a Hausdorff topological space with $\mathcal{B}(S/\rho) := \mathcal{B}(S)/\rho$ as its Borel sets. We infer from [3, Lemma 6.4.2 (i)] that $\mathcal{B}(S/\rho \otimes [0, 1]) = \mathcal{B}(S/\rho) \otimes \mathcal{B}([0, 1])$, because $[0, 1]$ is Polish. It is easy to see that $\Sigma_\rho \otimes \mathcal{B}([0, 1]) \subseteq \Sigma_{\rho \times \Delta}$, because each measurable rectangle $A \times B \in \Sigma_\rho \otimes \mathcal{B}([0, 1])$ is a $\rho \times \Delta$ -invariant measurable set, hence $A \times B \in \Sigma_{\rho \times \Delta}$. We claim that $\eta_{\rho \times \Delta}^{-1}[H] \in \Sigma_\rho \otimes \mathcal{B}([0, 1])$ for all $H \in \mathcal{B}(S/\rho) \otimes \mathcal{B}([0, 1])$.

In fact, let \mathcal{H} be the set of all elements of $\mathcal{B}(S/\rho) \otimes \mathcal{B}([0, 1])$ for which this is true. Then \mathcal{H} is a σ -algebra, and if $H = A \times B$ is a rectangle with $A \in \mathcal{B}(S/\rho)$, $B \in \mathcal{B}([0, 1])$, then

$$\eta_{\rho \times \Delta}^{-1}[H] = \eta_\rho^{-1}[A] \times B \in \Sigma_\rho \otimes \mathcal{B}([0, 1]).$$

Thus \mathcal{H} contains all measurable rectangles which generate the product σ -algebra, hence \mathcal{H} equals $\mathcal{B}(S/\rho) \otimes \mathcal{B}([0, 1])$. Now let $D \in \Sigma_{\rho \times \Delta}$, then

$$\eta_{\rho \times \Delta}[D] \in \mathcal{B}(S/\rho) \otimes \mathcal{B}([0, 1]) = \mathcal{B}(S/\rho \otimes [0, 1]),$$

hence

$$D = \eta_{\rho \times \Delta}^{-1}[\eta_{\rho \times \Delta}[D]] \in \Sigma_\rho \otimes \mathcal{B}([0, 1]).$$

This establishes the other inclusion. \square

This shows that tame equivalence relations constitute a generalization of smooth ones for the case that we do not work in a Polish environment. Smooth relations help in establishing interesting structural properties for stochastic relations, so it is to be expected that tame relations help in uncovering structural properties for the case of stochastic effectivity functions, to be discussed below. Specifically, we will deal with tame relations when we investigate congruences for stochastic effectivity functions.

3 Stochastic Effectivity Functions

The basic idea for an effectivity function is to produce upon some input s all the results which can be achieved through s . The results are modelled as subsets of some result set, so one of the basic requirements is that the family of sets thus achieved is upward closed: If A is a set which can be achieved, and $A \subseteq A'$, then it should be possible to achieve A' as well. Since we are in the realm of probabilities, we do not work directly with possible outcomes but rather with their distributions. So an effectivity function should produce an upward closed set of distributions over the space of outputs upon an input.

Associating the use of effectivity function with an action, the sequential composition of actions becomes important. Hence we want to be able to characterize the results achieved after the execution of two actions in sequence. This requires taking intermediate results into account: executing action γ will achieve certain results which then will be fed into the effectivity function associated with action δ , yielding then the overall result for executing $\gamma; \delta$. Again, because we are working in a probabilistic scenario, we will want to be able to average over intermediate results. This in turn requires a notion of measurability, permitting quantitative assessments.

These considerations, which have been formulated in the Introduction as requirements, will now be made specific. They lead to the introduction of stochastic effectivity functions over two measurable spaces. The latter can be interpreted as input resp. as output space, but when applying effectivity functions to game logic, the spaces coincide, forming the state space of the model under consideration. In order to be better able to distinguish the rôles of the domain and the range spaces, however, we separate both spaces for the purposes of the present paper.

Effectivity functions are also compared to non-deterministic Markov processes. We state elementary properties and investigate the relationship to stochastic relations, i.e., to the transition probabilities which are at the center of stochastic processes in probability theory, and which form the basis for stochastic Kripke models [6, 9, 28].

Put for the measurable space (T, \mathcal{T})

$$\mathbb{V}(T) := \{V \subseteq \mathcal{w}(\mathcal{T}) \mid V \text{ is upward closed}\}$$

thus if $V \in \mathbb{V}(T)$, then V is a collection of measurable sets of subprobabilities on the measurable space (T, \mathcal{T}) , moreover $A \in V$ and $A \subseteq B$ together imply $B \in V$ (for the definition of $\mathcal{w}(\mathcal{T})$ see (6)). A measurable map $f : S \rightarrow T$ induces a map $\mathbb{V}(f) : \mathbb{V}(S) \rightarrow \mathbb{V}(T)$ upon setting

$$(\mathbb{V}f)(V) := \mathbb{V}(f)(V) := \{W \in \mathcal{w}(\mathcal{T}) \mid \mathbb{S}(f)^{-1}[W] \in V\}$$

for $V \in \mathbb{V}(S)$, hence clearly $(\mathbb{V}f)(V) \in \mathbb{V}(T)$.

Note that $\mathbb{V}(T)$ has not been equipped with a σ -algebra, so the usual notion of measurability between measurable spaces cannot be applied; this is in contrast to the non-deterministic labeled Markov processes studied in [5].

To be specific, [5, Definition 3.1] defines for a given measurable space (S, \mathcal{S}) of states and a set L of labels a *non-deterministic labeled Markov process (NLMP)* as a family $(\kappa_a)_{a \in L}$

of measurable maps $\kappa_a : S \rightarrow \mathcal{W}(S)$. The target space $\mathcal{W}(S)$ is endowed with the smallest σ -algebra \mathcal{R}_S which contains the sets $\{H_G \mid G \in \mathcal{W}(S)\}$, where

$$H_G := \{C \in \mathcal{W}(S) \mid C \cap G \neq \emptyset\}$$

is the set of all measurable sets of $\mathbb{S}(S)$ which *hit* the given Borel set $G \subseteq \mathbb{S}(S)$ (consequently, this σ -algebra is called the *hit- σ -algebra*). So, for example $C \in H_{\beta_S(A, > q)}$ iff C contains some $\mu \in \mathbb{S}(S)$ with $\mu(A) > q$. It is easy to see that the σ -algebra \mathcal{R}_S is generated also by the upward closed sets $U_G := \{C \in \mathcal{W}(S) \mid G \subseteq C\}$ for $G \in \mathcal{W}(S)$, so that $\kappa_a : S \rightarrow \mathcal{W}(S)$ is measurable iff $\kappa_a^{-1}[U_G] \in \mathcal{B}(S)$ holds, whenever $G \in \mathcal{W}(S)$. The interplay between NLMPs and stochastic effectivity functions is investigated in [15].

Concerning \mathbb{V} , a closer look shows that $\mathbb{V}(T)$ can be derived from the composition $(\mathbb{W} \circ \mathbb{S})(T)$ with

$$\mathbb{W}(X) := \{W \subseteq \mathcal{P}(X) \mid W \text{ is upward closed}\} \quad (10)$$

for the set X , restricting \mathbb{W} to upward closed subsets of σ -algebras. We define for the map $f : X \rightarrow Y$

$$\mathbb{W}(f)(W) := \{Z \in \mathcal{P}(Y) \mid f^{-1}[Z] \in W\}. \quad (11)$$

Put for $x \in X$, $F : X \rightarrow \mathbb{W}(Y)$

$$\begin{aligned} \xi_X(x) &:= \{W \subseteq X \mid x \in W\}, \\ F^*(W) &:= \{B \subseteq Y \mid \{x \in X \mid B \in F(x)\} \in W\}, \end{aligned}$$

then it is not difficult to show [13, Example 1.53] that $(\mathbb{W}, -, \xi)$ is a Kleisli triple [26], hence forms a monad.

Let $H \in \mathcal{W}(\mathcal{T}) \otimes \mathcal{B}([0, 1])$ be a measurable subset of $\mathbb{S}(T) \times [0, 1]$ indicating a *quantitative assessment of subprobabilities*. A typical example could be

$$\{(\mu, q) \mid \mu \in \beta_T(A, \geq q), 0 \leq q \leq 1\} = \{(\mu, q) \mid \mu(A) \geq q, 0 \leq q \leq 1\}$$

for some $A \in \mathcal{T}$, asking for all combinations of subprobabilities and reals such that the probability for the given set A of states or events do not lie below this value. Consider a map $P : S \rightarrow \mathbb{V}(T)$, fix some real q and consider the *horizontal section* $H_q = \{\mu \mid (\mu, q) \in H\}$ of H at q , viz., the set of all measures evaluated through q . We ask for all states s such that H_q is *effective for s* , i.e., $\{s \in S \mid H_q \in P(s)\}$. This set should be a measurable subset of S ; an NLMP will have this property after a simple transformation, see below and [15]. It turns out, however, that this is not enough, we also require the real components being captured through a measurable set as well — after all, the real component will be used to be averaged over later on, so it should behave decently. This idea is captured in the following definition.

Definition 3.1 *Call a map $P : S \rightarrow \mathbb{V}(T)$ t-measurable iff $\{(\mu, q) \mid H_q \in P(s)\} \in \mathcal{S} \otimes \mathcal{B}([0, 1])$ whenever $H \in \mathcal{W}(\mathcal{T}) \otimes \mathcal{B}([0, 1])$*

Thus if P is t-measurable then we know in particular that all pairs of states and numerical values indicating the effectivity of the evaluation of a measurable set $A \in \mathcal{T}$, i.e., the set $\{(\mu, q) \mid \beta_T(A, \bowtie q) \in P(s)\}$ is always a measurable subset of $\mathcal{S} \otimes [0, 1]$. This is so because we know that

$$\{(\mu, q) \mid \mu \in \beta_T(A, \bowtie q), 0 \leq q \leq 1\} = \{(\mu, q) \in \mathbb{S}(T) \times [0, 1] \mid \mu(A) \bowtie q\} \in \mathcal{W}(\mathcal{T}) \otimes \mathcal{B}([0, 1])$$

by Theorem 2.3.

This leads to the notion of a stochastic effectivity function.

Definition 3.2 *Given measurable spaces S and T , a stochastic effectivity function $P : S \rightarrow T$ from S to T is a t -measurable map $P : S \rightarrow \mathbb{V}(T)$.*

We say that, given $P : S \rightarrow T$, the set $P(s)$ comprises the *portfolio* for $s \in S$, and that each element of $P(s)$ can be *achieved* or is *effective through P in s* ; sometimes S may be thought to be a set of inputs, and the portfolio to be defined over the set of outputs which can be achieved. If S and T coincide, representing a set of states, $P(s)$ may be interpreted as indicating the set of all state distributions which can be achieved in state s .

The kinship to NLMPs is fairly obvious. Let $P : S \rightarrow S$ be a stochastic effectivity function on a state space S , then $P(s)$ is an upward closed *subset* of $\mathcal{w}(S)$ for any $s \in S$. If $(\kappa_a)_{a \in L}$ is a non-deterministic labeled Markov process on the same state space, then $\kappa_a(s)$ is an *element* of $\mathcal{w}(S)$. It can be made a map $\kappa_a^* : S \rightarrow \mathbb{V}(S)$ upon setting

$$\kappa_a^*(s) := \{A \in \mathcal{w}(S) \mid \kappa_a(s) \subseteq A\}.$$

It is not difficult to see that the set $\{s \in S \mid H \in \kappa_a^*(s)\}$ is a member of \mathcal{S} whenever $H \in \mathcal{w}(S)$, since $\kappa_a : S \rightarrow \mathcal{w}(S)$ is \mathcal{S} - \mathcal{R}_S -measurable. But this does not suffice for our purposes, since a quantitative assessment is missing. Hence the kinship is remote only. These concepts are obviously similar in spirit and intention, viz., to capture stochastic non-determinism.

Let us have a look at some examples.

Example 3.3 We show that a finite transition system can be converted into a stochastic effectivity function.

Let $S := \{1, \dots, n\}$ for some $n \in \mathbb{N}$, and take the power set as a σ -algebra. Then $\mathbb{S}(S)$ can be identified with the compact convex set

$$\Pi_n := \{\langle x_1, \dots, x_n \rangle \mid x_i \geq 0 \text{ for } 1 \leq i \leq n, \sum_{i=1}^n x_i \leq 1\}.$$

Geometrically, Π_n is the convex hull of the unit vectors e_i , $1 \leq i \leq n$ and the zero vector; here $e_i(i) = 1$, and $e_i(j) = 0$ if $i \neq j$ is the i -th n -dimensional unit vector. The weak- $*$ - σ -algebra is the Borel- σ -algebra $\mathcal{B}(\Pi_n)$ for the Euclidean topology on Π_n .

Assume we have a transition system \rightarrow_S on S , hence a relation $\rightarrow_S \subseteq S \times S$. Put $\text{succ}(s) := \{s' \in S \mid s \rightarrow_S s'\}$ as the set of a successor states for state s . Define for $s \in S$ the set of weighted successors

$$\kappa(s) := \left\{ \sum_{s' \in \text{succ}(s)} \alpha_{s'} \cdot e_{s'} \mid \mathbb{Q} \ni \alpha_{s'} \geq 0 \text{ for } s' \in \text{succ}(s), \sum_{s' \in \text{succ}(s)} \alpha_{s'} \leq 1 \right\}$$

and the upward closed set

$$P(s) := \{A \in \mathcal{B}(\Pi_n) \mid \kappa(s) \subseteq A\}$$

A set A is in the portfolio for P in state s if A contains all rational distributions on the successor states. We will restrict our attention to these rational distributions, which are

positive convex combinations of the unit vectors with rational coefficients. Note that states may get lost, since we work with subprobabilities.

We claim that P is an effectivity function on S . If $P(s) = \emptyset$, there is nothing to show, so we assume that always $P(s) \neq \emptyset$. Let $H \in \mathcal{B}(\Pi_n) \otimes \mathcal{B}([0, 1]) = \mathcal{B}(\Pi_n \otimes [0, 1])$, then

$$\{\langle s, q \rangle \mid H_q \in P(s)\} = \bigcup_{1 \leq s \leq n} \{s\} \times \{q \in [0, 1] \mid H_q \in P(s)\}.$$

Fix $s \in S$, and let $\text{succ}(s) = \{s_1, \dots, s_m\}$. Put

$$\Omega_m := \{\langle \alpha_1, \dots, \alpha_m \rangle \in \mathbb{Q}^m \mid \alpha_i \geq 0, \sum_i \alpha_i \leq 1\}, \quad (12)$$

hence Ω_m is countable, and

$$\begin{aligned} \{q \in [0, 1] \mid H_q \in P(s)\} &= \{q \in [0, 1] \mid k(s) \subseteq H_q\} \\ &= \bigcap_{\langle \alpha_1, \dots, \alpha_m \rangle \in \Omega_m} \{q \in [0, 1] \mid \sum_i \alpha_i \cdot e_{j_i} \in H_q\}. \end{aligned}$$

Now fix $\alpha := \langle \alpha_1, \dots, \alpha_m \rangle \in \Omega_m$. The map $\zeta_\alpha : [0, 1]^{m \cdot n} \rightarrow [0, 1]^n$ which maps $\langle v_1, \dots, v_m \rangle$ to $\sum_{i=1}^m \alpha_i \cdot v_i$ is continuous, hence measurable, so is $\xi := \zeta_\alpha \times \text{id}_{[0, 1]} : [0, 1]^{m \cdot n} \times [0, 1] \rightarrow [0, 1]^n \times [0, 1]$. Hence $I := \xi^{-1}[H] \in \mathcal{B}([0, 1]^{m \cdot n} \times [0, 1])$, and $\sum_{i=1}^m \alpha_i \cdot e_{j_i} \in H_q$ iff $\langle e_{j_1}, \dots, e_{j_m}, q \rangle \in I$. Consequently,

$$\{q \in [0, 1] \mid \sum_i \alpha_i \cdot e_{j_i} \in H_q\} = I^{\langle e_{j_1}, \dots, e_{j_m} \rangle} \in \mathcal{B}([0, 1]).$$

But this implies that

$$\{q \in [0, 1] \mid H_q \in P(s)\} = \bigcap_{\alpha \in \Omega_m} \{q \in [0, 1] \mid \sum_i \alpha_i \cdot e_{j_i} \in H_q\} \in \mathcal{B}([0, 1])$$

for the fixed state $s \in S$. Collecting states, we obtain

$$\{\langle s, q \rangle \in S \times [0, 1] \mid H_q \in P(s)\} \in \mathcal{P}(S) \otimes \mathcal{B}([0, 1]).$$

Thus we have converted a finite transition system into a stochastic effectivity function by constructing all subprobabilities over the respective successor sets, albeit with rational coefficients. It is fairly easy to see that κ forms an NLMP.

One might ask whether the restriction to rational coefficients is really necessary. Taking the convex closure with real coefficients might, however, results in loosing measurability, see [24, p. 216]. —

The next example shows that a stochastic effectivity function can be used for interpreting a simple modal logic.

Example 3.4 Let Φ be a set of atomic propositions, and define the formulas of a logic through this grammar

$$\varphi ::= \top \mid p \mid \varphi_1 \wedge \varphi_2 \mid \Diamond_q \varphi$$

with $p \in \Phi$ an atomic proposition and $q \in [0, 1]$ a threshold value. Intuitively, $\Diamond_q \varphi$ is true in a state s iff there can be a move in s to a state in which φ holds with probability not smaller than q .

This logic is interpreted over a measurable space (S, \mathcal{S}) ; assume that we are given a map $e : \Phi \rightarrow \mathcal{S}$, assigning each atomic proposition a measurable set as its validity set. Let $P : S \rightarrow \mathcal{S}$ be a stochastic effectivity function over (S, \mathcal{S}) , then define inductively

$$\begin{aligned} \llbracket \top \rrbracket &:= S, \\ \llbracket p \rrbracket &:= e(p), \text{ for } p \in \Phi, \\ \llbracket \varphi_1 \wedge \varphi_2 \rrbracket &:= \llbracket \varphi_1 \rrbracket \cap \llbracket \varphi_2 \rrbracket, \\ \llbracket \Diamond_q \varphi \rrbracket &:= \{s \in S \mid \beta_S(\llbracket \varphi \rrbracket, > q) \in P(s)\}. \end{aligned}$$

The interesting line is of course the last one. It assigns to $\Diamond_q \varphi$ all states s such that $\beta_S(\llbracket \varphi \rrbracket, > q)$ is in the portfolio of $P(s)$. These are all states for which the collection of all measures yielding an evaluation on $\llbracket \varphi \rrbracket$ greater than q can be achieved.

Then t -measurability of P and the assumption on e make sure that these sets are measurable. This is shown by induction on the structure of the formulas. —

We have a look now at stochastic relations as a concept which specializes both NLMPs and stochastic effectivity functions.

Example 3.5 Let $K : S \rightsquigarrow T$ be a stochastic relation, then

$$P_K(s) := \{A \in \mathcal{w}(\mathcal{T}) \mid K(s) \in A\}$$

defines a stochastic effectivity function $P_K : S \rightarrow \mathcal{T}$ [11, Lemma 4.4]. Similarly, assume $\mathcal{F} = \{K_n \mid n \in \mathbb{N}\}$ is a countable family of stochastic relations $K_n : S \rightsquigarrow T$, then

$$\begin{aligned} (\exists \mathcal{F})(s) &:= \{A \in \mathcal{w}(\mathcal{T}) \mid K_n(s) \in A \text{ for some } n \in \mathbb{N}\}, \\ (\forall \mathcal{F})(s) &:= \{A \in \mathcal{w}(\mathcal{T}) \mid K_n(s) \in A \text{ for all } n \in \mathbb{N}\} \end{aligned}$$

define stochastic effectivity functions $S \rightarrow \mathcal{T}$. These functions resemble the weak resp. the strong inverse of set-valued maps studied in topology or economics [25, 20]. —

A pattern seems to arise here: take a hit-measurable map $\kappa : S \rightarrow \mathcal{w}(\mathcal{T})$, and define

$$P_\kappa(s) := \{A \in \mathcal{w}(\mathcal{T}) \mid \kappa(s) \subseteq A\}.$$

Then $P_\kappa(s)$ is evidently upward closed, and the example above suggests that this yields a stochastic effectivity function. It turns out, however, that this picture has to be scrutinized carefully. If $\kappa(s)$ is always finite, then P_κ is a stochastic effectivity function indeed, i.e., P_κ is t -measurable. If, however, $\kappa(s)$ is uncountable for some s , t -measurability is gone forever. This is discussed in some detail in [15].

Example 3.6 Continuing Example 3.5, let $\{1, \dots, \ell\}$ be a set of individuals, and K_1, \dots, K_ℓ be a finite set of stochastic relations. $K_j : S \rightsquigarrow T$ is intended to model the preferences of individual j . Then the set of all rational positive convex combinations (Ω_ℓ as in (12))

$$\mathcal{F} := \left\{ \sum_{j=1}^{\ell} \alpha_j \cdot K_j \mid \langle \alpha_1, \dots, \alpha_\ell \rangle \in \Omega_\ell \right\}$$

defines a countable family of stochastic relations $S \rightsquigarrow T$. The relation $\sum_{j=1}^{\ell} \alpha_j \cdot K_j$ indicates a possible group preference, individual j being assigned weight α_j (the weights do not necessarily

add up to 1, indicating some possible loss along the process of coordination). If we have $A \in (\exists\mathcal{F})(s)$, then the set A of distributions indicates that portfolio A is possible in state s for certain preferences $\alpha \in \Omega_\ell$, if however $A \in (\forall\mathcal{F})(s)$, portfolio A is always preferred. This idea can be extended easily to a countable number of individuals with finite coalitions by taking a sequence $(K_n)_{n \in \mathbb{N}}$ of stochastic relations and considering

$$\Omega' := \{(\alpha_n)_{n \in \mathbb{N}} \mid \sum_n \alpha_n \leq 1, 0 \leq \alpha_n \text{ rational}, \alpha_n \neq 0 \text{ for finitely many } n\}$$

rather than Ω_ℓ . The same *caveat* as in Example 3.3 applies when it comes to taking the “full” convex closure with real coefficients. —

Effectivity functions which are generated through stochastic relations can be characterized in terms of principal ultrafilters, as the next proposition shows.

Proposition 3.7 *Let $P : S \rightarrow \mathcal{T}$ be a stochastic effectivity function. Then these statements are equivalent*

1. $P(s)$ is a principal ultrafilter on $\mathcal{W}(\mathcal{T})$ for each $s \in S$.
2. $P = P_K$ for some stochastic relation $K : S \rightsquigarrow \mathcal{T}$.

Proof We shown only 1 \Rightarrow 2: Let $P(s)$ be a principal ultrafilter for each $s \in S$, and define

$$\{K(s)\} := \bigcap P(s).$$

Then $K(s) \in \mathcal{S}(\mathcal{T})$ for each $s \in S$, and we show now that $\{s \in S \mid K(s)(G) > q\} \in \mathcal{S}$ for each $G \in \mathcal{T}, 0 \leq q \leq 1$. We claim first that $K(s) \in \beta_{\mathcal{T}}(G, > q)$ is equivalent to $\beta_{\mathcal{T}}(G, > q) \in P(s)$. For assume that $K(s) \in \beta_{\mathcal{T}}(G, > q)$ but $\beta_{\mathcal{T}}(G, > q) \notin P(s)$. Since $P(s)$ is an ultrafilter, $\mathcal{S}(\mathcal{T}) \setminus \beta_{\mathcal{T}}(G, > q) = \beta_{\mathcal{T}}(G, \leq q) \in P(s)$. But this means $K(s)(G) \leq q$, contradicting the assumption. Now we infer

$$\{\langle s, q \rangle \mid K(s)(G) > q\} = \{\langle s, q \rangle \mid \beta_{\mathcal{T}}(G, > q) \in P(s)\} \in \mathcal{S} \otimes \mathcal{B}([0, 1]),$$

which implies that we have in particular $\{s \in S \mid K(s)(G) > q\} \in \mathcal{S}$ for fixed $q \in [0, 1]$. \square

The relationship of stochastic effectivity functions and stochastic relations can be characterized also through an approach resembling deduction systems [17, 11]. This is sketched here, the reader is referred to [11, Section 4.1] for details and proofs.

A *characteristic relation* $R \subseteq [0, 1] \times \mathcal{T}$ on the measurable space $(\mathcal{T}, \mathcal{T})$ is defined through these conditions

$\frac{\langle r, A \rangle \in R, A \subseteq B}{\langle r, B \rangle \in R}$ $\frac{\langle r, A \rangle \notin R, \langle s, B \rangle \notin R, r + s \leq 1}{\langle r + s, A \cup B \rangle \notin R}$ $\frac{\langle r, A \rangle \in R, r + s > 1}{\langle s, S \setminus A \rangle \notin R}$ $\frac{A_1 \supseteq A_2 \supseteq \dots, \forall n \in \mathbb{N} : \langle r, A_n \rangle \in R}{\langle r, \bigcap_{n \geq 1} A_n \rangle \in R}$	$\frac{\langle r, A \rangle \in R, r \geq s}{\langle s, A \rangle \in R}$ $\frac{\langle r, A \cup B \rangle \in R, \langle s, A \cup (S \setminus B) \rangle \in R, r + s \leq 1}{\langle r + s, A \rangle \in R}$ $\frac{\langle r, \emptyset \rangle \in R}{r = 0}$
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Such a characteristic relation R defines a subprobability $\mu_R \in \mathbb{S}(T)$ through

$$\mu_R(B) := \sup\{r \in [0, 1] \mid \langle r, B \rangle \in R\}.$$

An upper closed set $Q \in \mathbb{V}(T)$ is said to *satisfy* relation R iff we have

$$\langle q, A \rangle \in R \Leftrightarrow \beta_T(A, \geq q) \in Q,$$

and Q is said to *implement* $\mu \in \mathbb{S}(T)$ iff

$$\mu(A) \geq q \Leftrightarrow \beta_T(A, > q) \in Q.$$

Then Q satisfies the characteristic relation R iff it implements μ_R . Moreover, for a given effectivity function $P : S \rightarrow T$ there exists a stochastic relation $K : S \rightsquigarrow T$ with $P = P_K$ iff

$$R(s) := \{\langle r, B \rangle \mid \beta_T(B, \geq r) \in P(s)\}$$

defines for each $s \in S$ a characteristic relation such that $R(s)$ satisfies $P(s)$.

3.1 Morphisms

Stochastic effectivity functions P and Q can be compared through morphisms, which are based on measurable functions. Roughly speaking, a set is in the portfolio of $Q(f(s))$ iff its inverse image can be achieved in $P(s)$. This idea is made precise. Having formulated what morphisms are, we compare them with morphisms for stochastic relations. Congruences are also defined in terms of morphisms, see Section 3.4.

Fix the measurable spaces (S, \mathcal{S}) , (T, \mathcal{T}) , (U, \mathcal{U}) and (V, \mathcal{V}) for the rest of the present paper.

Given stochastic effectivity functions $P : S \rightarrow T$ and $Q : U \rightarrow V$, a pair of measurable maps $f : S \rightarrow U$ and $g : T \rightarrow V$ is called a *morphism of effectivity functions* $(f, g) : P \rightarrow Q$ iff this diagram of maps commutes

$$\begin{array}{ccc} S & \xrightarrow{f} & U \\ P \downarrow & & \downarrow Q \\ \mathbb{V}(T) & \xrightarrow{\mathbb{V}g} & \mathbb{V}(V) \end{array}$$

Thus we have

$$W \in Q(f(s)) \Leftrightarrow (\mathbb{S}g)^{-1}[W] \in P(s) \quad (13)$$

for all states $s \in S$ and for all $W \in \mathcal{W}(U)$; hence the set W of distributions is in the portfolio of $Q(f(s))$ iff its inverse image is in the portfolio of $P(s)$. Technically, this definition derives from the one for functor \mathbb{W} , see (11).

Let us compare this to morphisms for stochastic relations. A pair of measurable maps $f : S \rightarrow U$ and $g : T \rightarrow V$ is a *morphism of stochastic relations* $(f, g) : K \rightarrow L$ for the stochastic

relations $K : S \rightsquigarrow T$ and $L : U \rightsquigarrow V$ iff this diagram commutes

$$\begin{array}{ccc} S & \xrightarrow{f} & U \\ \downarrow K & & \downarrow L \\ S(T) & \xrightarrow{\mathbb{S}g} & S(V) \end{array}$$

Thus $L(f(s)) = (\mathbb{S}g)(K(s))$, which means

$$L(f(s))(B) = (\mathbb{S}g)(K(s))(B) (= K(s)(g^{-1}[B])) \quad (14)$$

for each state $s \in S$ and each measurable set $B \in \mathcal{V}$. This says that the L -distributions of states for $f(s)$ is just the K -distribution of states for s , transformed by $\mathbb{S}g$.

These notions of morphisms are related to each other: Each morphism for stochastic relations turns into a morphism for the associated effectivity function (we will usually do without the attributions to effectivity functions or stochastic relations when talking about morphisms, whenever the context is clear).

Proposition 3.8 *A morphism $(f, g) : K \rightarrow L$ for stochastic relations K and L induces a morphism $(f, g) : P_K \rightarrow P_L$ for the associated stochastic effectivity functions.*

Proof Fix a state $s \in S$. Then $W \in P_L(f(s))$ iff $L(f(s)) \in W$. Because (f, g) is a morphism $K \rightarrow L$, this is equivalent to $(\mathbb{S}g)(K(s)) \in W$, hence to $K(s) \in (\mathbb{S}g)^{-1}[W]$, thus $(\mathbb{S}g)^{-1}[W] \in P_K(s)$. \square

This result suggests that stochastic effectivity functions are an algebraically meaningful generalization of stochastic relations.

3.2 Convolutions

Given a stochastic effectivity function $P : S \rightarrow T$, we ask for all states $s \in S$ such that $P(s)$ contains a given Borel subset D with a probability not smaller than a threshold value q . When we bind P to an action γ , this question corresponds to asking for all states that permit to observe a given effect not below probability q (e.g., in a game logic, one might ask for all states such that a formula holds with at least probability q upon playing a specified game in that state). This leads to a map $\mathcal{T} \times [0, 1] \rightarrow \mathcal{S}$.

This map is investigated, and we show that it is possible to model the sequential composition of actions through this construction; here the assumption on t -measurability pays off. We give a quick comparison to the composition of stochastic relations through the Kleisli product and show how convolution is related to it.

Let $P : S \rightarrow T$ be a stochastic effectivity function, then

$$P^b(D, q) := \{s \in S \mid \beta_T(D, > q) \in P(s)\}$$

defines a map P^b on $\mathcal{T} \times [0, 1]$ into the power set of S . Since $P(s)$ is upward closed for each $s \in S$, the map $P^b(\cdot, q)$ is increasing for each q . Now let $D \in \mathcal{T}$, then

$$H := \{\langle \mu, r \rangle \mid \mu \in \beta_T(D, > r)\} = \{\langle \mu, r \rangle \mid \mu(D) > r\}$$

is a member of $\mathcal{w}(\mathcal{T}) \otimes \mathcal{B}([0, 1])$ by Choquet's Theorem, thus

$$P^b(D, q) = (\{\langle s, r \rangle \mid H_r \in P(s)\})_q \in \mathcal{S}.$$

Hence P^b maps $\mathcal{T} \times [0, 1]$ to \mathcal{S} .

Intuitively, $P^b(D, q)$ gives all states from which it is possible to achieve a portfolio exceeding the given threshold q . Binding an action γ to P in the sense that $P(s)$ is the set of all distributions over T upon action γ which can be achieved in state s , we interpret $P^b(D, q)$ as the set of all states s for which we can achieve a state in $D \in \mathcal{T}$ with a probability greater than q .

Let $Q : T \rightarrow \mathcal{U}$ be another stochastic effectivity function with associated map $Q^b : \mathcal{U} \times [0, 1] \rightarrow \mathcal{T}$. Define the *convolution* of P^b and Q^b through

$$(P^b * Q^b)(E, q) := \{s \in S \mid G_Q(E, q) \in P(s)\},$$

where

$$G_Q(E, q) := \{\nu \in \mathbb{S}(T) \mid \int_0^1 \nu(Q^b(E, r)) \, dr > q\}.$$

Binding action γ to P and δ to Q , respectively, the effect of executing the combined sequential action $\gamma; \delta$, i.e., first γ and then δ , is modeled through $P^b * Q^b$.

Formally, $P^b * Q^b$ yields a map from $\mathcal{U} \times [0, 1]$ to \mathcal{S} which is monotone in its first component:

Proposition 3.9 *Let $P : S \rightarrow T$ and $Q : T \rightarrow \mathcal{U}$ be stochastic effectivity functions, then $P^b * Q^b$ maps $\mathcal{U} \times [0, 1]$ to \mathcal{S} such that $E \mapsto (P^b * Q^b)(E, q)$ is increasing for all q .*

Proof 1. We show first that

$$G'_Q(E) := \{\langle \nu, q \rangle \mid \int_0^1 \nu(Q^b(E, r)) \, dr > q\} \in \mathcal{w}(\mathcal{T}) \otimes \mathcal{B}([0, 1]),$$

whenever $E \in \mathcal{U}$. In order to apply Choquet's Theorem 2.3 for showing that this set is measurable, we write

$$G'_Q(E) = \{\langle \nu, q \rangle \in \mathbb{S}(T) \times [0, 1] \mid F(\nu) > q\}$$

with

$$\begin{aligned} F(\nu) &:= \int_0^1 \nu(Q^b(E, r)) \, dr \\ &= \int_0^1 \nu(\{t \in T \mid \beta_{\mathcal{U}}(E, > r) \in Q(t)\}) \, dr \\ &= (\nu \otimes \lambda)(\{(t, r) \in T \times [0, 1] \mid \beta_{\mathcal{U}}(E, > r) \in Q(t)\}) \end{aligned}$$

by Theorem 2.3; here $\nu \otimes \lambda$ is the product measure on $\mathcal{T} \otimes \mathcal{B}([0, 1])$ with factors ν and the Lebesgue measure λ . Then we have to show that F depends measurably on ν . Hence it suffices to show that $\nu \mapsto (\nu \otimes \lambda)(G)$ is a $\mathcal{w}(\mathcal{T})$ - $\mathcal{B}([0, 1])$ -measurable map for each $G \in \mathcal{T} \otimes \mathcal{B}([0, 1])$.

2. Let

$$\mathcal{G} := \{G \in \mathcal{T} \otimes \mathcal{B}([0, 1]) \mid \nu \mapsto (\nu \otimes \lambda)(G) \text{ is } \mathcal{w}(\mathcal{T})\text{-}\mathcal{B}([0, 1])\text{-measurable}\}.$$

Then \mathcal{G} is a σ -algebra. This is clear from the familiar properties of measurable maps. Moreover, \mathcal{G} contains $D \times B$ for $D \in \mathcal{T}, B \in \mathcal{B}([0, 1])$. This follows from the definition of the weak- $*$ - σ -algebra as the smallest σ -algebra which renders evaluating measures measurable, and because $(\nu \otimes \lambda)(D \times B) = \nu(B) \cdot \lambda(B)$. Consequently, \mathcal{G} equals $\mathcal{T} \otimes \mathcal{B}([0, 1])$.

3. Because

$$\{(\mathbf{t}, r) \in T \times [0, 1] \mid \beta_U(E, > r) \in Q(\mathbf{t})\} \in \mathcal{T} \otimes \mathcal{B}([0, 1])$$

by the definition of \mathbf{t} -measurability, we infer that F is measurable, hence $G'_Q(E) \in \mathcal{w}(\mathcal{T}) \otimes \mathcal{B}([0, 1])$ by Choquet's Theorem. But because $G_Q(E, q) = (G'_Q(E))_q$, we conclude from \mathbf{t} -measurability of P that $(P^b * Q^b)(E, q) \in \mathcal{S}$, hence $P^b * Q^b : \mathcal{U} \times [0, 1] \rightarrow \mathcal{S}$. Monotonicity is obvious. \square

Let us have a look at the behavior of stochastic relations in this scenario. Given the stochastic relations $K : S \rightsquigarrow T$ and $L : T \rightsquigarrow U$, the *convolution* (or the *Kleisli product*) $K * L : S \rightsquigarrow U$ of K and L is defined through

$$(K * L)(s)(V) := \int_T L(t)(V) K(s)(dt).$$

Again, binding an action \mathbf{a} to K , $K(s)(V)$ is interpreted as the probability of reaching an element of V upon executing action \mathbf{a} in state s ; binding \mathbf{b} to L , the probability $(K * L)(s)(U)$ is interpreted as the probability for the combined action $\mathbf{a}; \mathbf{b}$.

Converting a stochastic relation to a stochastic effectivity function is compatible with convolutions:

Lemma 3.10 *Let $K : S \rightsquigarrow T$ and $L : T \rightsquigarrow U$ be stochastic relations with associated effectivity functions $P_K : S \twoheadrightarrow T$ resp. $P_L : T \twoheadrightarrow U$. Then*

$$P_K^b * P_L^b = P_{K * L}^b.$$

Proof We obtain for $F \in \mathcal{U}, 0 \leq r \leq 1$ by expanding definitions

$$P_L^b(F, r) = \{\mathbf{t} \in T \mid \beta_U(F, > r) \in P_L(\mathbf{t})\} = \{\mathbf{t} \in T \mid L(\mathbf{t})(F) > r\},$$

thus

$$\int_0^1 \nu(P_L^b(F, r)) \, dr > q \Leftrightarrow \int_T L(\mathbf{t})(F) \, \mu(d\mathbf{t}) > q.$$

Now define

$$\Gamma_{P_L}(F) := \{(\mu, q) \mid \int_0^1 \mu(P_L^b(F, r)) \, dr > q\},$$

then

$$\begin{aligned}
P_{K*L}^b(F, q) &= \{s \in S \mid \beta_U(F, > q) \in P_{K*L}(s)\} \\
&= \{s \in S \mid (K * L)(s)(F) > q\} \\
&= \{s \in S \mid \{\mu \in \mathbb{S}(T) \mid \int_T L(t)(F) \mu(dt) > q\} \in P_K(s)\} \\
&= \{s \in S \mid (\Gamma_{P_L}(F))_q \in P_K(s)\} \\
&= (P_K^b * P_L^b)(F, q)
\end{aligned}$$

□

Hence the convolution of stochastic relations finds its counterpart in the convolution of the monotone maps which are induced by an effectivity functions. Thus stochastic effectivity functions may be used for modelling sequentiality, similar to the use of stochastic relations when modelling, e.g., dynamic logics.

3.3 Induced monotone maps

The map P^b is actually obtained as a special case. Define for $P : S \rightarrow T$ and for $H \in w(\mathcal{T}) \otimes \mathcal{B}([0, 1])$ the set

$$\mathfrak{J}(P)(H) := \mathfrak{p}_P(H) := \{\langle s, q \rangle \mid H_q \in P(s)\},$$

as all pairs of states and numeric values for which H can be achieved, then

$$\mathfrak{p}_P : w(\mathcal{T}) \otimes \mathcal{B}([0, 1]) \rightarrow \mathcal{S} \otimes \mathcal{B}([0, 1])$$

by the definition of t -measurability. The map \mathfrak{p} is monotone, and clearly

$$P^b(D, q) = (\mathfrak{p}_P(\{\langle \mu, r \rangle \in \mathbb{S}(T) \times [0, 1] \mid \mu \in \beta_T(D, > r)\}))_q.$$

This correspondence goes even a bit deeper when considering morphisms. Let $\mathfrak{p} : w(\mathcal{T}) \otimes \mathcal{B}([0, 1]) \rightarrow \mathcal{S} \otimes \mathcal{B}([0, 1])$ and $\mathfrak{p}' : w(\mathcal{V}) \otimes \mathcal{B}([0, 1]) \rightarrow w(\mathcal{U}) \otimes \mathcal{B}([0, 1])$ be monotone maps, and take measurable maps $f : S \rightarrow U$ and $g : T \rightarrow V$. Then $(f, g) : \mathfrak{p} \rightarrow \mathfrak{p}'$ is said to be a *morphism* iff

$$\langle f(s), q \rangle \in \mathfrak{p}'(H') \Leftrightarrow \langle s, q \rangle \in \mathfrak{p}((\mathbb{S}(g) \times \text{id}_{[0,1]})^{-1}[H'])$$

holds for all $s \in S$ and for all $q \in [0, 1]$. This condition is equivalent to saying that

$$(f \times \text{id}_{[0,1]})^{-1}[\mathfrak{p}'(H')] = \mathfrak{p}((\mathbb{S}(g) \times \text{id}_{[0,1]})^{-1}[H'])$$

holds for all $H' \in w(\mathcal{V}) \otimes \mathcal{B}([0, 1])$. Consequently, this diagram of maps commutes

$$\begin{array}{ccc}
w(\mathcal{T}) \otimes \mathcal{B}([0, 1]) & \xleftarrow{(\mathbb{S}(g) \times \text{id}_{[0,1]})^{-1}} & w(\mathcal{V}) \otimes \mathcal{B}([0, 1]) \\
\mathfrak{p} \downarrow & & \mathfrak{p}' \downarrow \\
\mathcal{S} \otimes \mathcal{B}([0, 1]) & \xleftarrow{(f \times \text{id}_{[0,1]})^{-1}} & \mathcal{U} \otimes \mathcal{B}([0, 1])
\end{array}$$

The relationship between these morphisms is fairly transparent, as we will show now.

Proposition 3.11 *Let $P : S \rightarrow T$ and $Q : U \rightarrow V$ be stochastic effectivity functions, and assume $f : S \rightarrow U$ and $g : T \rightarrow V$ are measurable. Then $(f, g) : P \rightarrow Q$ is a morphism iff $(f, g) : \mathcal{I}(P) \rightarrow \mathcal{I}(Q)$ is a morphism.*

Proof 0. Let $p := \mathcal{I}(P)$, $q := \mathcal{I}(Q)$ for easier notation. We have to show

$$(f, g) : P \rightarrow Q \Leftrightarrow (f, g) : p \rightarrow q.$$

1. “ \Rightarrow ” Observe that we have

$$\langle \mu, q \rangle \in (\mathbb{S}(g) \times \text{id}_{[0,1]})^{-1}[H'] \Leftrightarrow \mu \in \mathbb{S}(g)^{-1}[H'_q]$$

for $H' \in w(\mathcal{V}) \otimes \mathcal{B}([0, 1])$, thus

$$\begin{aligned} \langle s, q \rangle \in p((\mathbb{S}(g) \times \text{id}_{[0,1]})^{-1}[H']) &\Leftrightarrow ((\mathbb{S}(g) \times \text{id}_{[0,1]})^{-1}[H'])_q \in P(s) \\ &\Leftrightarrow \mathbb{S}(g)^{-1}[H'_q] \in P(s) \stackrel{(\dagger)}{\Leftrightarrow} H'_q \in Q(f(s)) \Leftrightarrow \langle f(s), q \rangle \in q(H') \\ &\Leftrightarrow \langle s, q \rangle \in (f \times \text{id}_{[0,1]})^{-1}[q(H')] \end{aligned}$$

Here (\dagger) uses the assumption that (f, g) is a morphism for effectivity functions.

2. “ \Leftarrow ” Now let $W \in Q(f(s))$, for $s \in S$, so that we can find $H' \in w(\mathcal{V}) \otimes \mathcal{B}([0, 1])$ and $q \in [0, 1]$ with $W = H'_q$ and $\langle f(s), q \rangle \in q(H')$. Because

$$\begin{aligned} \langle s, q \rangle \in (f \times \text{id}_{[0,1]})^{-1}[q] &\stackrel{(\dagger)}{\Leftrightarrow} \langle s, q \rangle \in p((\mathbb{S}(g) \times \text{id}_{[0,1]})^{-1}[H']) \\ &\Leftrightarrow ((\mathbb{S}(g) \times \text{id}_{[0,1]})^{-1}[H'])_q \in P(s) \\ &\Leftrightarrow \mathbb{S}(g)^{-1}[H'_q] \in P(s) \\ &\Leftrightarrow \mathbb{S}(g)^{-1}[W] \in P(s) \end{aligned}$$

(using the assumption in (\dagger)) we find that $Q(f(s)) = \mathbb{V}(g)(P(s))$, establishing the claim. \square

Thus morphisms for stochastic effectivity functions are in natural bijective correspondence with morphisms for monotone maps. For plain effectivity functions between sets, this correspondence is easily established and silently used for the interpretation of game logic. Adding quantitative information and working with distributions rather than with states renders these relationships somewhat more complicated, but exhibits a very similar structure.

3.4 Congruences

Let $P : S \rightarrow T$ be a fixed stochastic effectivity function.

Congruences are defined as usual through morphisms and factorization. Because we define effectivity functions between two spaces, a congruence will have to capture properties of both spaces, so in this general setting a congruence is a pair. The idea is that if two elements s, s' of S cannot be separated through the equivalence on S , then it should not be possible to separate the portfolios of $P(s)$ and $P(s')$ through the corresponding equivalence on $\mathbb{S}(T)$. For expressing this adequately, we require an effectivity function on the respective factor spaces which is compatible with the factor structure. This leads to the following definition.

Definition 3.12 A congruence for P is a pair (α, β) of equivalence relations on S resp. T such that there exists an effectivity function $P_{(\alpha, \beta)} : S/\alpha \rightarrow T/\beta$ which renders this diagram commutative

$$\begin{array}{ccc} S & \xrightarrow{\eta_\alpha} & S/\alpha \\ P \downarrow & & \downarrow P_{(\alpha, \beta)} \\ \mathbb{V}(T) & \xrightarrow{\mathbb{V}\eta_\beta} & \mathbb{V}(T/\beta) \end{array}$$

If $c = (\alpha, \beta)$ is a congruence for P , the effectivity function $P_{(\alpha, \beta)}$ is also denoted by P/c .

Consequently, we have

$$W \in P_{(\alpha, \beta)}([s]_\alpha) \Leftrightarrow (\mathbb{S}\eta_\beta)^{-1}[W] \in P(s)$$

for $W \in \mathcal{w}(\mathcal{T}/\beta)$ and $s \in S$. So if $s \alpha s'$, we have in particular $(\mathbb{S}\eta_\beta)^{-1}[W] \in P(s)$ iff $(\mathbb{S}\eta_\beta)^{-1}[W] \in P(s')$, which means that $P(s)$ and $P(s')$ cannot separate those portfolios which are indistinguishable under $\mathbb{S}\eta_\beta$.

Because η_α is onto, $P_{(\alpha, \beta)}$ is uniquely determined. The next proposition provides a criterion for an equivalence relation to be a congruence. It requires the equivalence relation α to be tame.

Proposition 3.13 Given a stochastic effectivity function $P : S \rightarrow T$ and equivalence relations α on S and β on T with α tame, these statements are equivalent

1. (α, β) is a congruence for P .
2. Whenever $s \alpha s'$, we have $(\mathbb{S}\eta_\beta)^{-1}[A] \in P(s)$ iff $(\mathbb{S}\eta_\beta)^{-1}[A] \in P(s')$ for every $A \in \mathcal{w}(\mathcal{T}/\beta)$

Proof “1 \Rightarrow 2”: This follows immediately from the definition.

“2 \Rightarrow 1”: Define for $s \in S$

$$P_{(\alpha, \beta)}([s]_\alpha) := \{A \in \mathcal{w}(\mathcal{T}/\beta) \mid (\mathbb{S}\eta_\beta)^{-1}[A] \in P(s)\},$$

then $P_{(\alpha, \beta)}$ is well defined by the assumption, and it is clear that $P_{(\alpha, \beta)}([s]_\alpha)$ is an upward closed set of subsets of $\mathcal{w}(\mathcal{T}/\beta)$ for each $s \in S$. It remains to show that $P_{(\alpha, \beta)}$ is a stochastic effectivity function, i.e., that $P_{(\alpha, \beta)}$ is t -measurable.

In fact, let $H \in \mathcal{w}(\mathcal{T}/\beta) \otimes \mathcal{B}([0, 1])$ be a test set, and put

$$Y := \{ \langle [s]_\alpha, q \rangle \mid H_q \in P_{(\alpha, \beta)}([s]_\alpha) \} = (\eta_\alpha \times \text{id}_{[0, 1]})[Z]$$

with

$$Z := \{ \langle s, q \rangle \mid (\mathbb{S}\eta_\beta)^{-1}[H_q] \in P(s) \}$$

as its inverse image under $\mathbb{S}(\eta_\alpha) \times \text{id}_{[0, 1]}$. Then Z is $(\alpha \times \Delta)$ -invariant. By Corollary 2.11 it is enough to show that Z is a member of $\Sigma_\alpha \otimes \mathcal{B}([0, 1])$. Then $Y \in \mathcal{S}/\alpha \otimes [0, 1]$ will follow.

Because P is t -measurable, we infer $Z \in \mathcal{S} \otimes \mathcal{B}([0, 1])$. Since Z is $(\alpha \times \Delta)$ -invariant, we conclude from $\mathbb{S}(\eta_\beta)^{-1}[H_q] = ((\mathbb{S}(\eta_\beta) \times \text{id}_{[0, 1]})^{-1}[H])_q$ and $(\mathbb{S}(\eta_\beta) \times \text{id}_{[0, 1]})^{-1}[H] \in \mathcal{w}(\mathcal{T}) \otimes \mathcal{B}([0, 1])$

that $Z \in \Sigma_{\alpha \times \Delta}$. The latter σ -algebra is equal to $\Sigma_\alpha \otimes \mathcal{B}([0, 1])$ by Lemma 2.10, because α is tame. \square

As a consequence, the kernel of a morphism (f, g) is a congruence, provided $\ker(f)$ is tame, and provided g is final. The proof uses the observation formulated in Corollary 2.6 that equivalence relations induced by the kernel of a final and surjective measurable map preserve the σ -algebras on which they operate.

Proposition 3.14 *Let $(f, g) : P \rightarrow Q$ be a morphism for the stochastic effectivity functions $P : S \twoheadrightarrow T$ and $Q : U \twoheadrightarrow V$. If $f \times \text{id}_{[0,1]}$ and g are final, then $(\ker(f), \ker(g))$ is a congruence for P .*

Proof 1. If $f \times \text{id}_{[0,1]}$ is final, then $\ker(f \times \text{id}_{[0,1]}) = \ker(f) \times \Delta$ is tame by Lemma 2.5. Hence it remains to show that condition 2 in Proposition 3.13 is satisfied.

2. Decompose $g = \tilde{g} \circ \eta_{\ker(g)}$ with $\tilde{g} : T/\ker(g) \rightarrow V$ (see the proof of Lemma 2.5), and put

$$\mathcal{Z} := \{A \in \mathcal{w}(\mathcal{T}/\ker(g)) \mid A = (\mathbb{S}\tilde{g})^{-1}[A_1] \text{ for some } A_1 \in \mathcal{w}(\mathcal{V})\}.$$

Then \mathcal{Z} is a σ -algebra; we show that $\mathcal{Z} = \mathcal{w}(\mathcal{T}/\ker(g))$. Let $G \in \mathcal{T}/\ker(g)$, then finality of g implies that we find $G_1 \in \mathcal{V}$ with $G = \tilde{g}^{-1}[G_1]$, as in the proof of Lemma 2.5, thus

$$\beta_{T/\ker(g)}(G, > q) = \beta_{T/\ker(g)}(\tilde{g}^{-1}[G_1], > q) = \mathbb{S}(\tilde{g})^{-1}[\beta_U(G_1, > q)]$$

Consequently,

$$\{\beta_{T/\ker(g)}(G, > q) \mid G \in \mathcal{T}/\ker(g)\} \subseteq \mathcal{Z}.$$

Since the former set generates $\mathcal{w}(\mathcal{T}/\ker(g))$, the claim follows.

3. Now let $B \in \mathcal{w}(\mathcal{T}/\ker(g))$, hence we find $B_1 \in \mathcal{w}(\mathcal{T})$ with $\mathbb{S}(\eta_{\ker(g)})^{-1}[B] = \mathbb{S}(g)^{-1}[B_1]$, thus we obtain for $\langle s, s' \rangle \in \ker(f)$

$$\begin{aligned} \mathbb{S}(\eta_{\ker(g)})^{-1}[B] \in P(s) &\Leftrightarrow \mathbb{S}(g)^{-1}[B_1] \in P(s) \\ &\Leftrightarrow B_1 \in Q(f(s)) = Q(f(s')) \\ &\Leftrightarrow \mathbb{S}(\eta_{\ker(g)})^{-1}[B] \in P(s'), \end{aligned}$$

because (f, g) is a morphism. \square

A morphism $(f, g) : P \rightarrow Q$ is called *strong* iff $f \times \text{id}_{[0,1]}$ and g are final; a congruence (α, β) is called *tame* iff α is a tame equivalence relation. Thus we have shown that the kernel of a strong morphism is a tame congruence, and vice versa.

The correspondence between strong morphisms and tame equivalence relations will turn out to be fairly tight, as we will see when investigating logical and behavioral equivalence of stochastic effectivity functions.

4 Logical and Behavioral Equivalence

Interpreting a logic, the question of expressivity of the underlying models is important. For example, logical equivalence requires that we find for each state in one model another state

which satisfies exactly the same formulas; through factoring we then will be able under some circumstances to build smaller models with the same expressive power. We are in a position now to characterize logical resp. behavioral equivalence through morphisms and congruences, and to relate these notion of expressivity in purely algebraic terms, i.e., without reference to an underlying logic. Behavioral equivalence is expressed through a co-span of surjective morphisms, and logical equivalence is expressed through isomorphic factor spaces. Investigating the relationship between the two, we show that logically equivalent effectivity functions are behaviorally equivalent, and that the converse holds as well, provided we assume that the morphisms are strong.

Given two stochastic effectivity functions $P : S \rightarrow T$ and $Q : U \rightarrow V$, call P and Q *logically equivalent* iff there exist tame congruences c for P and d for Q such that P/c and Q/d are isomorphic. The name derives from an observation for Kripke models in modal logics: two states are called equivalent iff they have the same theory, i.e., accept exactly the same formulas, and two Kripke models are called logically equivalent iff given a state in one model, there exists a state in the other one with the same theory. Then it can be shown for stochastic Kripke models over analytic spaces in a fairly general, coalgebraic context that the corresponding factor models are isomorphic [14, Corollary 4.8, Theorem 6.17].

Similarly, call P and Q *behaviorally equivalent* iff there exists a mediating function $M : X \rightarrow Y$ and strong surjective morphisms $(f, g) : P \rightarrow M$ and $(k, \ell) : Q \rightarrow M$. Thus we obtain the familiar diagram

$$\begin{array}{ccccc} S & \xrightarrow{f} & X & \xleftarrow{k} & U \\ P \downarrow & & M \downarrow & & \downarrow Q \\ \mathbb{V}(T) & \xrightarrow{\mathbb{V}g} & \mathbb{V}(Y) & \xleftarrow{\mathbb{V}\ell} & \mathbb{V}(V) \end{array} \quad (15)$$

This diagram translates into

$$D \in M(f(s)) \Leftrightarrow (\mathbb{S}g)^{-1}[D] \in P(s) \quad (16)$$

$$D \in M(k(u)) \Leftrightarrow (\mathbb{S}\ell)^{-1}[D] \in Q(u) \quad (17)$$

for $D \subseteq \mathbb{S}(Y)$ measurable and $s \in S, u \in U$.

The correspondence between the use of tame equivalence relations for logical equivalence and of strong morphisms for behavioral equivalence is noteworthy. From a technical point of view, tame relations are necessary for constructing the factor model. This suggests the use of strong morphisms, because then the corresponding kernels form a congruence (see Proposition 3.13). Thus we require a strong morphism in order to factor through the kernels, which in turn enables us to compare factor models.

We fix for the sequel the stochastic effectivity functions $P : S \rightarrow T$ and $Q : U \rightarrow V$.

Proposition 4.1 *If P and Q are logically equivalent, they are behaviorally equivalent.*

Proof If α is a tame equivalence relation, then $\eta_\alpha \times \text{id}_{[0,1]}$ is final. \square

In order to show that behaviorally equivalent effectivity functions are logically equivalent, fix $M : X \rightarrow Y$ and the morphisms according to diagram (15) with measurable spaces X and Y ; to make notation not heavier than it is, we do without an explicit name for the σ -algebras on X

resp. Υ . Recall that we simplify notations by writing, e.g., $[s]_f$ rather than $[s]_{\ker(f)}$, similarly for Σ_f .

$$\begin{aligned}\gamma &:= \{ \langle [s]_f, [u]_k \rangle \mid s \in S, u \in U, f(s) = k(u) \} \\ \delta &:= \{ \langle [t]_g, [v]_\ell \rangle \mid t \in T, v \in V, g(t) = \ell(v) \}\end{aligned}$$

Because the contributing maps are onto, γ and δ are the graphs of bijective maps; this is shown exactly as in the proof of [10, Lemma 2.6.10]. For simplicity, the maps proper are called δ and γ as well.

Lemma 4.2 $\gamma : S/\ker(f) \rightarrow U/\ker(k)$ and $\delta : T/\ker(g) \rightarrow V/\ker(\ell)$ are Borel isomorphisms.

Proof We show that $\gamma^{-1}[E] \in S/\ker(f)$ for $E \in U/\ker(k)$. Because by construction $\eta_k^{-1}[E] \in \Sigma_k$, and because k is final, we find a measurable subset $H \subseteq X$ with $\eta_k^{-1}[E] = k^{-1}[H]$. Put $G := f^{-1}[H]$, then $G \in \Sigma_f$. Thus $G_0 := \eta_f[G] \in S/\ker(f)$. So we are done, provided we can show that $\gamma^{-1}[E] = G_0$ holds.

In fact, let $[s]_f \in \gamma^{-1}[E]$ with $[u]_k := \gamma([s]_f) \in E$, thus $f(s) = k(u)$. Hence $u \in k^{-1}[H]$, which implies $f(s) \in H$, so that $s \in f^{-1}[H] = G$, which in turn means $[s]_f \in G_0$, so that $\gamma^{-1}[E] \subseteq G_0$. On the other hand, if $[s]_f \in G_0$, we know that $s \in G$. Hence $f(s) \in H$, so that we can find $u \in U$ with $f(s) = k(u)$. Since $k(u) \in H$, we know $u \in k^{-1}[H] = \eta_k^{-1}[E]$, but this means $[u]_k = \gamma([s]_f) \in E$, establishing the other inclusion.

The claim for δ is established in exactly the same way. \square

Because $(f, g) : P \twoheadrightarrow M$ is a strong morphism, its kernel $\ker(f, g) := (\ker(f), \ker(g))$ is a congruence by Proposition 3.14, similarly for $(k, \ell) : Q \twoheadrightarrow M$. Hence the factor functions $P/\ker(f, g) : S/\ker(f) \twoheadrightarrow T/\ker(g)$ and $Q/\ker(k, \ell) : U/\ker(k) \twoheadrightarrow V/\ker(\ell)$ exist.

It turns out that these functions yield isomorphisms.

Proposition 4.3 $(\gamma, \delta) : P/\ker(f, g) \rightarrow Q/\ker(k, \ell)$ is an isomorphism.

Proof 0. We know from Lemma 4.2 that both γ and δ are Borel isomorphisms, so we have to show that they are compatible with the structure of the effectivity functions. We show that this diagram commutes

$$\begin{array}{ccc} S/\ker(f) & \xrightarrow{\gamma} & U/\ker(k) \\ P/\ker(f, g) \downarrow & & \downarrow Q/\ker(k, \ell) \\ \mathbb{V}(T/\ker(g)) & \xrightarrow{\mathbb{V}\delta} & \mathbb{V}(V/\ker(\ell)) \end{array}$$

Interchanging the rôles of γ and δ will then establish the result.

1. Fix $G \in w(\mathbb{V}/\ell)$ and $s \in S$, we show that

$$G \in (Q/k, \ell)(\gamma([s]_f)) \Leftrightarrow (\mathbb{S}\delta)^{-1}[G] \in (P/f, g)([s]_f).$$

From Lemma 4.4 we infer that we can find for G a measurable set $H \subseteq \mathbb{S}(Y)$ such that

$$\mathbb{S}(\delta \circ \eta_g)^{-1}[G] = (\mathbb{S}g)^{-1}[H] \quad (18)$$

$$(\mathbb{S}\eta_\ell)^{-1}[G] = (\mathbb{S}\ell)^{-1}[H] \quad (19)$$

Given $s \in S$, we find $u \in U$ such that $f(s) = k(u)$. Then we have

$$\begin{aligned} (\mathbb{S}\delta)^{-1}[G] \in (P/f, g)([s]_f) &\Leftrightarrow (\mathbb{S}\eta_g)^{-1}[(\mathbb{S}\delta)^{-1}[G]] \in P(s) \\ &\stackrel{(18)}{\Leftrightarrow} (\mathbb{S}g)^{-1}[H] \in P(s) \stackrel{(16)}{\Leftrightarrow} H \in M(f(s)) = M(k(u)) \stackrel{(17)}{\Leftrightarrow} (\mathbb{S}\ell)^{-1}[H] \in Q(u) \\ &\stackrel{(19)}{\Leftrightarrow} (\mathbb{S}\eta_\ell)^{-1}[G] \in Q(u) \Leftrightarrow G \in (Q/k, \ell)(\gamma([s]_f)) \end{aligned}$$

This was to be shown. \square

We have delayed, however, the proof of an auxiliary statement.

Lemma 4.4 *For every $G \in w(\mathcal{V}/\ell)$ there exists a measurable set $H \subseteq \mathbb{S}(Y)$ such that*

$$\mathbb{S}(\delta \circ \eta_g)^{-1}[G] = (\mathbb{S}g)^{-1}[H] \text{ and } (\mathbb{S}\eta_\ell)^{-1}[G] = (\mathbb{S}\ell)^{-1}[H] \quad (20)$$

Proof 1. Let \mathcal{Z} be the set of all $G \in w(\mathcal{V}/\ell)$ such that the assertion is true. Then \mathcal{Z} is a σ -algebra, so that it is sufficient to demonstrate that the statement is true for a generator $G = \beta_{V/\ker(\ell)}(A, > q)$ with $A \in \mathcal{V}/\ker(\ell)$ and $0 \leq q \leq 1$.

2. Because $A \in \mathcal{V}/\ker(\ell)$, we know that $A = \eta_\ell[A_0]$ for some $A_0 \in \Sigma_\ell$. Thus $A_0 = \ell^{-1}[H_0]$ for some measurable set $H_0 \subseteq Y$. We claim that $H := \beta_Y(H_0, > q)$ is the set we are looking for in (20).

3. First we note that $A_0 = \eta_\ell^{-1}[A]$, hence

$$\begin{aligned} (\mathbb{S}\ell)^{-1}[H] &= (\mathbb{S}\ell)^{-1}[\beta_Y(H_0, > q)] = \beta_V(\ell^{-1}[H_0], > q) = \beta_V(A_0, > q) \\ &= \beta_V(\eta_\ell^{-1}[A], > q) = (\mathbb{S}\eta_\ell)^{-1}[\beta_{V/\ell}(A, > q)] = (\mathbb{S}\eta_\ell)^{-1}[G]. \end{aligned}$$

4. Then we claim that

$$t \in g^{-1}[H_0] \Leftrightarrow \delta([t]_g) \in \eta_\ell[\ell^{-1}[H_0]] \quad (21)$$

Assume that $g(t) \in H_0$, and take $v \in V$ with $g(t) = \ell(v)$, thus $\delta([t]_g) = [v]_\ell$, hence $[v]_\ell \in \eta_\ell[\ell^{-1}[H_0]]$. On the other hand, if $\delta([t]_g) \in \eta_\ell[\ell^{-1}[H_0]]$, there exists $v \in V$ such that $g(t) = \ell(v)$, so that $\ell(v) \in H_0$, which implies $t \in g^{-1}[H_0]$.

Thus we obtain

$$\begin{aligned} (\mathbb{S}g)^{-1}[H] &= (\mathbb{S}g)^{-1}[\beta_Y(H_0, > q)] = \beta_T(g^{-1}[H_0], > q) \\ &\stackrel{(21)}{=} \beta_T(\eta_g^{-1}[\delta^{-1}[\eta_\ell[\ell^{-1}[H_0]]]], > q) = \mathbb{S}(\delta \circ \eta_g)[\beta_{V/\ker(f)}(\eta_\ell[\ell^{-1}[H_0]], > q)] \\ &= \mathbb{S}(\delta \circ \eta_g)[\beta_{V/\ker(f)}(\eta_\ell[A_0], > q)] = \mathbb{S}(\delta \circ \eta_g)[\beta_{V/\ker(f)}(A, > q)] = \mathbb{S}(\delta \circ \eta_g)[G] \end{aligned}$$

Consequently, H is the set we are looking for.

5. This implies

$$\{\beta_{V/\ker(\ell)}(A, > q) \mid A \in \mathcal{V}/\ker(\ell), 0 \leq q \leq 1\} \subseteq \mathcal{Z},$$

so that $\mathcal{Z} = w(\mathcal{V}/\ell)$. □

Summarizing, we have shown

Proposition 4.5 *Behaviorally equivalent stochastic effectivity functions are logically equivalent.* □

Thus we observe a close correspondence of behavioral and logical equivalence, which might be compared to a similar result obtained in [8] for stochastic relations. There these equivalences are compared with each other and with bisimilarity, and under fairly strong topological assumptions it was shown that a result similar to Proposition 4.5 can be obtained. We do not impose topological assumptions in the present paper, but rather require the morphisms to be strong, and the equivalence relations to be tame, so there is a trade off among assumptions. On the other hand, Proposition 3.7 tells us that stochastic effectivity functions are strictly more general than stochastic relations.

5 Conclusion

The algebraic properties of stochastic effectivity functions have been studied, and we have investigated the question of expressivity of these functions. Logical and behavioral equivalence have been related to each other; this has been done without recourse to an underlying logic.

Bisimilarity is usually a companion to logical and to behavioral equivalence. Call two stochastic effectivity functions P and Q *bisimilar* iff there is a mediating function M to which P and Q are related through a span $P \leftarrow M \rightarrow Q$ of morphisms. This is the coalgebraic definition of bisimilarity. In the context of effectivity functions for games, a relational definition is used, quite close to Milner's original one [2, 31]. It was shown, however, that the relational and the coalgebraic one are equivalent [13, Proposition 1.142], provided the relation's projections are surjective (see also [18]).

While it is possible to relate logical and behavioral equivalence algebraically for general measurable spaces, this seems to be more involved for the case of bisimilarity. In a comparable situation for stochastic relations, it was shown that bisimilar relations are logically equivalent under a compactness assumption, the converse was established for Polish spaces through a selection argument [8]; see [14] for a broader survey.

Related questions remain to be looked at carefully as well, among them the relationship to non-deterministic labeled Markov processes, in particular the question to determine under which conditions such a process is generated by a stochastic effectivity function. Certainly, t -measurability plays a crucial rôle; [15] discusses some of these issues, introduces subsystems, and proposes a partial solution to the problem of bisimilar effectivity functions, as expected, under some topological conditions.

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